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Research Article

The Existence and Application of Unbounded Connected Components

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Let X be a Banach space and C_n a family of connected subsets of $R \times X$. We prove the existence of unbounded components in superior limit of $\{C_n\}$, denoted by $\overline{\lim}\, C_n$, which have prescribed shapes. As applications, we investigate the global behavior of the set of positive periodic solutions to nonlinear first-order differential equations with delay, which can be used for modeling physiological processes.

1. Introduction and the Main Results

The connectivity result on the fixed set of a 1-parameter family of maps, which goes back to Leray and Schauder [1] and was proved in its full generality by Browder [2], is a useful tool in the study of global continua of solutions on nonlinear differential equations. Costa and Gonçalves [3] stated and proved a suitable version for the study of nonlinear boundary value problems at resonance. Massabò and Pejsachowicz [4] generalized the main results of [1, 2] to the n-parameter family of compact vector fields. The above results were established when the parameter(s) changes in a bounded set. Sun and Song [5] proved the existence of unbounded connected component of 1-parameter family of compact vector fields, where the parameter varies on whole real line. All of these results play important roles in the study of nonlinear functional analysis and nonlinear differential equations.

For clearly reading, we firstly recall Kuratowski's definitions and notations in [6].

Let $\mathbb{M}$ be a metric space. Let $\{C_n \mid n = 1, 2, \ldots\}$ be a family of subsets of $\mathbb{M}$. Then the *superior limit* $\mathscr{D}$ of $\{C_n\}$ is defined by

$$\mathscr{D} := \overline{\lim}\, C_n = \Big\{x \in M \mid \exists \{n_i\} \subset \mathbb{N},\ x_{n_i} \in C_{n_i}, \text{ such that } x_{n_i} \longrightarrow x\Big\}. \tag{1}$$

A *component* of a set $\mathbb{M}$ means a maximal connected subset of $\mathbb{M}$.

Definition 1. Let X be a Banach space with the norm $\|\cdot\|$. Let ζ be a component of solutions in $\mathbb{R} \times X$. ζ meets $(a, \mathbf{0})$ and infinity means that there existed a sequence $\{(\lambda_k, u_k)\} \subset [\zeta \setminus \{(a, \mathbf{0})\}]$ such that $(\lambda_k, u_k) \to (a, \mathbf{0})$ as $k \to \infty$.

For $\rho, \beta \in (0, \infty)$, let us denote

$$B_\rho := \{u \in X \mid \|u\| \le \rho\},$$
$$\Omega_{\beta,\rho} := ([0,\infty) \times X) \setminus \{(\eta, u) \in [\beta, \infty) \times X \mid \|u\| \le \rho\}. \tag{2}$$

Let $\{\mathscr{C}_n\}$ be a family of connected subsets of $\mathbb{R} \times X$. The purpose of this paper is to study the existence of unbounded components in $\overline{\lim}\, \mathscr{C}_n$ which have prescribed shapes.

More precisely, we will prove the following theorems.

Theorem 2. *Let X be a Banach space and let $\{\mathscr{C}_n\}$ be a family of connected subsets of $[0, \infty) \times X$. Assume that*

(A1) *there exist $0 < \sigma < r < \infty$ and $\lambda_* \in (0, \infty)$, such that*

$$\mathscr{C}_n \cap \{(\mu, y) \mid 0 < \mu \le \lambda_* + \sigma, r - \sigma \le \|y\| \le r + \sigma\} = \emptyset; \tag{3}$$

(A2) $\eta_n \to 0^+$ *and $\mathscr{C}_n$ meets $(\eta_n, \mathbf{0})$ and infinity;*

(A3) *for every* $R > 0$, $(\bigcup_{n=1}^{\infty} \mathscr{C}_n) \bigcap \mathbb{B}_R$ *is a relatively compact set of* $\mathbb{R} \times X$, *where*

$$\mathbb{B}_R = \{(\mu, x) \in \mathbb{R} \times X \mid |\mu| < R,\ \|x\| \le R\}. \tag{4}$$

Then there exists a component $\mathscr{C}$ *in* $\overline{\lim}\, \mathscr{C}_n$ *satisfying*

(a) $\mathscr{C}$ *meets* $(0, \mathbf{0})$ *and infinity;*

(b) $\mathscr{C} \cap \{(\mu, y) \mid 0 < \mu \le \lambda_*, \|y\| = r\} = \emptyset$.

Theorem 3. *Let* $a \in \mathbb{R}$ *be a constant. Let* X *be a Banach space, and let* $\{\mathscr{C}_n\}$ *be a family of connected subsets of* $\mathbb{R} \times X$. *Assume that*

(H1) $\mathscr{C}_n \cap ((-\infty, a] \times X) = \emptyset$;

(H2) *there exist* $0 < \sigma < r < \infty$ *and* $b \in (a, \infty)$, *such that*

$$\mathscr{C}_n \cap \{(\mu, y) \mid \mu \ge b - \sigma,\ r - \sigma \le \|y\| \le r + \sigma\} = \emptyset; \tag{5}$$

(H3) $\eta_k > a$ *for all* $k \in \mathbb{N}$, $\eta_k \to +\infty$ *and* $\mathscr{C}_n$ *meets* $(\eta_n, \mathbf{0})$ *and infinity in* $([a, \infty) \times X) \setminus \Omega_{b,r}$;

(H4) *for every* $R > 0$, $(\bigcup_{n=1}^{\infty} \mathscr{C}_n) \cap \mathbb{B}_R$ *is a relatively compact set of* $\mathbb{R} \times X$.

Then there exists a component $\mathscr{C}$ *in* $\overline{\lim}\, \mathscr{C}_n$ *such that*

(a) *both* $\mathscr{C} \cap \Omega_{b,r}$ *and* $\mathscr{C} \cap (([a, \infty) \times X) \setminus \Omega_{b,r})$ *are unbounded;*

(b) $\mathscr{C} \cap \{(\mu, y) \mid \mu \ge b, \|y\| = r\} = \emptyset$.

2. Proofs of the Main Results

To prove Theorems 2 and 3, we need the following preliminary result, which is proved by Ma and An [7] by using Whyburn lemma, and the method of Sun and Song to prove [6, Lemma 2.2].

Lemma 4 (see [7, Lemma 2.2]). *Let* E *be a Banach space with the norm* $\|\cdot\|_E$. *Let* $\{D_n\}$ *be a family of connected subsets of* E. *Assume that*

(i) *there exist* $z_n \in D_n$, $n = 1, 2, \ldots$, *and* $z^* \in E$, *such that* $z_n \to z^*$;

(ii) $\lim_{n\to\infty} r_n = \infty$, *where* $r_n = \sup\{\|x\|_E \mid x \in D_n\}$;

(iii) *for every* $R > 0$, $(\bigcup_{n=1}^{\infty} D_n) \cap B_R$ *is a relatively compact set of* E, *where*

$$B_R = \{x \in E \mid \|x\|_E \le R\}. \tag{6}$$

Then there exists an unbounded component $\mathscr{C}$ *in* $\overline{\lim}\, D_n$ *and* $z^* \in \mathscr{C}$.

Proof of Theorem 2. (a) It is a direct consequence of Lemma 4.

(b) Assume, on the contrary, that the conclusion is not true. Then there exists $(\mu^*, u^*) \in \mathscr{C}$ with $\mu^* \le \lambda_*$ and $\|u^*\| = r$. Hence, there exists $\{(\eta_{n_k}, u_{n_k})\} \subset \mathscr{C}_n$, such that

$$\lim_{k\to\infty} \eta_{n_k} = \mu^*, \qquad \lim_{k\to\infty} u_{n_k} = u^*. \tag{7}$$

Thus, there exists $k_0 \in \mathbb{N}$, such that, for $k \ge k_0$,

$$\eta_{n_k} < \lambda^* + \frac{\sigma}{2}, \qquad r - \frac{\sigma}{2} \le \|u_{n_k}\| \le r + \frac{\sigma}{2}. \tag{8}$$

However, this contradicts (3). □

Proof of Theorem 3. (a) Since $\eta_k \to \infty$, we may assume that

$$\eta_k > b, \quad k \in \mathbb{N}. \tag{9}$$

So, it follows from conditions (ii) and (iii) that $\mathscr{C}_n$ meets $\{b\} \times B_r$ and infinity in $([a, \infty) \times X) \setminus \Omega_{b,r}$.

For each $(b, v) \in (\{b\} \times B_r) \cap \mathscr{D}$, let $\mathscr{E}(b, v) (\subset \mathscr{D})$ be a component containing (b, v). Let

$$\varsigma(b, v) := \sup\{\lambda \mid (\lambda, u) \in \mathscr{E}(b, v),\ u \in B_r\}. \tag{10}$$

Set

$$\Pi := \{(b, v) \mid (b, v) \in (\{b\} \times B_r) \cap \mathscr{D},\ \mathscr{E}(b, v) \text{ is unbounded in } ([a, \infty) \times X) \setminus \Omega_{b,r}\}. \tag{11}$$

Then $\Pi \ne \emptyset$ since

$$(\mathscr{C}_j \cap (\{b\} \times B_r)) \subseteq \Pi, \quad j \in \mathbb{N}. \tag{12}$$

From Lemma 4, it follows that Π is closed in $[0, \infty) \times X$, and, furthermore, Π is compact in $[0, \infty) \times X$.

Let

$$\Sigma^{\diamond} := \bigcup_{(b,v)\in\Pi} \mathscr{E}(b, v). \tag{13}$$

By Lemma 4, $\overline{\lim}\, C_n \cap (([a, \infty) \times X) \setminus \Omega_{b,r})$ contains a component ζ which meets $\{b\} \times B_r$ and infinity in $(([a, \infty) \times X) \setminus \Omega_{b,r})$. Obviously

$$\zeta \in \Sigma^{\diamond}. \tag{14}$$

If $\varsigma(b, v) = +\infty$ for some $(b, v) \in \Pi$, then Theorem 3 holds.

Assume, on the contrary, that $\varsigma(b, v) < +\infty$ for all $(b, v) \in \Pi$.

For every $(b, v) \in \Pi$, let $\mathscr{E}^{\triangleright}(b, v)$ be the component in $\mathscr{E}(b, v) \cap ([b, \infty) \times B_r)$ which contains (b, v). Using the standard method, we can find a bounded open set $U(b, v)$ in $[b, \infty) \times B_r$, such that

$$\mathscr{E}^{\triangleright}(b, v) \subset U(b, v), \qquad \partial U(v) \cap \Sigma^{\diamond} = \emptyset \tag{15}$$

and for every $(b, v) \in \Pi$,

$$\sup\{\lambda \mid (\lambda, u) \in \overline{U}(b, v)\} < \infty, \tag{16}$$

where $\partial U(v)$ and $\overline{U}(v)$ are the boundary and closure of $U(v)$ in $[b, \infty) \times B_r$, respectively.

Evidently, the following family of the open sets of $\{b\} \times B_r$

$$\{U(b, v) \cap (\{b\} \times B_r) \mid (b, v) \in \Pi\} \tag{17}$$

is an open covering of Π. Since Π is compact set in $\{b\} \times B_r$, there exist $v_1, \ldots, v_m$ such that $(b, v_i) \in \Pi$ $(i = 1, \ldots, m)$, and the family of open sets in $\{b\} \times B_r$:

$$\{U(b, v_i) \cap (\{b\} \times B_r) \mid i = 1, \ldots, m\} \tag{18}$$

is a finite open covering of Π. This implies that

$$\Pi \subseteq \{U(b, v_i) \cap (\{b\} \times B_r) \mid i = 1, \ldots, m\}. \tag{19}$$

Let

$$U_1 = \bigcup_{i=1}^{m} U(b, v_i). \tag{20}$$

Then U_1 is a bounded open set in $[b, \infty) \times B_r$,

$$\partial U_1 \cap \left(\bigcup_{(b,v)\in\Pi} \mathscr{C}^{\triangleright}(b, v) \right) = \emptyset, \qquad \partial U_1 \cap \Sigma^{\diamond} = \emptyset \tag{21}$$

and by (16), we have

$$\sup\left\{\lambda \mid (\lambda, u) \in \overline{U}_1\right\} < \infty, \tag{22}$$

where ∂U_1 and $\overline{U}_1$ are the boundary and closure of U_1 in $[b, \infty) \times B_r$, respectively.

Now, (22) together with (19) and (21) implied that

$$\sup\left\{\lambda \mid (\lambda, u) \in \Sigma^{\diamond},\ u \in B_r\right\} < \infty. \tag{23}$$

However, this contradicts $\lambda_n \to \infty$.

Therefore, there exists $(b, v^*) \in \Pi$ such that $\zeta := \mathscr{C}(b, v^*)$ which is unbounded in both $[b, \infty) \times B_r$ and $([a, \infty) \times X) \setminus ([b, \infty) \times B_r)$.

(b) By a fully analogous argument as in the proof of Theorem 2(b) (with minor modifications), one can immediately obtain the desired results. □

3. Application to Functional Differential Equations

In recent years, there has been considerable interest in the existence of ω-periodic solutions of the equation

$$u'(t) = a(t)\, g(u(t))\, u(t) - \lambda h(t)\, f(u(t - \tau(t))), \quad t \in \mathbb{R}, \tag{24}$$

where $a, h \in C(\mathbb{R}, [0, \infty))$ are ω-periodic functions and τ is a continuous ω-periodic function. Equation (24) has been proposed as a model for a variety of physiological processes and conditions including production of blood cells, respiration, and cardiac arrhythmias. See, for example, [8–20] and the references therein.

Recently, Wang [18] used the fixed point index [20, 21] to study the existence, multiplicity, and nonexistence of positive solutions of (24) under the following assumptions.

(C1) $a, h \in C(\mathbb{R}, [0, \infty))$ are ω-periodic functions, $\int_0^{\omega} a(t)\, dt > 0$, $h(t) > 0$ on $[0, \omega]$; $\tau \in C(\mathbb{R}, \mathbb{R})$ is ω-periodic functions.

(C2) $f, g : [0, \infty) \to [0, \infty)$ are continuous. $0 < l \le g(s) \le L$ for $s > 0$, l, L are given positive constants. $f(s) > 0$ for $s > 0$.

Let $\sigma := e^{-\int_0^{\omega} a(t)dt}$ and denote

$$M(r) := \max_{0 \le t \le r} \{f(t)\},$$
$$m(r) := \min\left\{ f(t) : \frac{\sigma^L (1 - \sigma^l)}{1 - \sigma^L} r \le t \le r \right\}; \tag{25}$$
$$f_0 := \lim_{u \to 0^+} \frac{f(u)}{u}, \qquad f_\infty := \lim_{u \to +\infty} \frac{f(u)}{u}.$$

His results provide no any information about the global behavior of the set of positive solutions of (24).

In this section, we will use Theorems 2 and 3 to establish several results on the global behavior of the set of positive solutions of (24), and, accordingly, we get some existence and multiplicity results of positive solutions of (24).

We will work essentially in the Banach space $X = \{u \in C[0, \omega] \mid u$ is ω-periodic.$\}$ with sup norm $\|\cdot\|$.

By a positive solution of (24), we mean a pair (λ, u), where $\lambda > 0$ and u is a solution of (24) with $u > 0$ on $[0, \omega]$.

Let $\Sigma \subset \mathbb{R}^+ \times E$ be the closure of the set of positive solutions of (24).

We extend the function f to a continuous function $\tilde{f}$ defined on $\mathbb{R}$ in such a way that $\tilde{f} > 0$ for all $s < 0$. For $\lambda > 0$, we then look at arbitrary solutions u of the eigenvalue problem

$$-u'(t) + a(t)\, g(u(t))\, u(t) = \lambda h(t)\, \tilde{f}(u(t - \tau(t))), \quad t \in \mathbb{R},\ u \text{ is } \omega\text{-periodic}. \tag{26}$$

It was shown in [18] that (26) is equivalent to

$$u(t) = \lambda \int_t^{t+\omega} G_u(t, s)\, h(s)\, \tilde{f}(u(s - \tau(s)))\, ds, \tag{27}$$

where

$$G_u(t, s) = \frac{e^{-\int_t^s a(\theta) g(u(\theta)) d\theta}}{1 - e^{-\int_0^{\omega} a(\theta) g(u(\theta)) d\theta}}, \qquad t \le s \le t + \omega. \tag{28}$$

By the positivity of Green's function $G_u(\cdot, \cdot)$, $h(\cdot)$, and $f(\cdot)$, such solutions are positive. Therefore, the closure of the set of nontrivial solutions (λ, u) of (24) in $\mathbb{R}^+ \times X$ is exactly Σ.

Next, we consider the spectrum of the linear eigenvalue problem

$$-u'(t) + a(t)\, c u(t) = \lambda h(t)\, u(t - \tau(t)), \quad t \in \mathbb{R},\ u \text{ is } \omega\text{-periodic}. \tag{29}$$

Lemma 5. *Let c be a positive constant. Then the linear problem (29) has a unique eigenvalue $\lambda^{\diamond}(c)$, which is positive and simple, and the corresponding eigenfunction $\psi(\cdot)$ is of one sign.*

Proof. Define an operator $T : X \to X$ by

$$Tu(t) = \lambda \int_t^{t+\omega} G_u(t, s)\, h(s)\, \tilde{f}(u(s - \tau(s)))\, ds. \tag{30}$$

Let K be the cone

$$K = \{u \in X \mid u \geq 0\}. \tag{31}$$

Then it follows from [18, Lemma 2.2] that $T : K \to K$ is strongly positive and completely continuous. Thus, the desired result is a direct consequence of Krein-Rutman Theorem, cf. [22, Theorem 19.3]. □

Theorem 6. *Let (C1)-(C2) hold. Assume that*

(C3) $f_0 = \infty = f_\infty$.

Then Σ contains a component $\mathscr{C}$ satisfying

(1) *$\mathscr{C}$ meets $(0, \mathbf{0})$ and $(0, \infty)$;*

(2) *for every $r > 0$, there exists $\mu(r) > 0$, such that*

$$\mathscr{C} \cap \{(\mu, u) \in (0, \mu(r)) \times X \mid \|u\| = r\} = \emptyset. \tag{32}$$

Corollary 7. *Let (C1)–(C3) hold. Then there is a constant $\eta_* > 0$ such that (24) has at least two positive solutions as $0 < \lambda < \eta_*$.*

Proof. Let

$$\begin{aligned} B &:= \sup\{\lambda \mid (\lambda, u) \in \mathscr{C}\}, \\ \eta_* &:= \sup\{\mu(r) \mid r \in (0, \infty)\}. \end{aligned} \tag{33}$$

Then $0 < \eta_* \leq B$. It is easy to see from Theorem 6 that (24) has at least two positive solutions as $0 < \lambda < \eta_*$. □

Denote the cone K in X by

$$\begin{gathered} K = \{u \in X \mid u(t) \geq 0 \text{ on } [0, \omega], \ u(t) \geq \delta \|u\|\}, \\ \delta := \frac{\sigma^L (1 - \sigma^l)}{1 - \sigma^L}, \end{gathered} \tag{34}$$

and for $r > 0$, let

$$K_r = \{u \in K \mid \|u\| < r\}. \tag{35}$$

Define an operator $T_\lambda : K \to X$ by

$$T_\lambda u(t) = \lambda \int_t^{t+\omega} G_u(t, s) h(s) \tilde{f}(u(s - \tau(s))) \, ds, \quad t \in [0, \omega]. \tag{36}$$

Lemma 8 (see [18]). *Assume that (C1)-(C2) hold. Then $T_\lambda : K \to K$ is completely continuous.*

Lemma 9. *Let (C1)-(C2) hold. If $u \in \partial K_r$, $r > 0$, then*

$$\|T_\lambda u\| \leq \lambda \frac{M_r}{1 - \sigma^l} \int_0^\omega h(s) \, ds, \tag{37}$$

where $M_r = 1 + \max_{\delta r \leq s \leq r}\{f(s)\}$.

Proof. It is well known from Wang [18] that

$$\frac{\sigma^L}{1 - \sigma^L} \leq G_u(t, s) \leq \frac{1}{1 - \sigma^l}, \quad t \leq s \leq t + \omega. \tag{38}$$

This together with the fact that h is ω-periodic and $u(t) \geq \delta r$ on $[0, \omega]$ implies that

$$\|T_\lambda u\| \leq \lambda \frac{M_r}{1 - \sigma^l} \int_t^{t+\omega} h(s) \, ds = \lambda \frac{M_r}{1 - \sigma^l} \int_0^\omega h(s) \, ds. \tag{39}$$

□

To prove above Theorem 6, we define $f^{[n]} : [0, \infty) \to [0, \infty)$ by

$$f^{[n]}(s) = \begin{cases} f(s), & s \in \left(\dfrac{1}{n}, \infty\right), \\ nf\left(\dfrac{1}{n}\right) s, & s \in \left[0, \dfrac{1}{n}\right]. \end{cases} \tag{40}$$

Then $f^{[n]} \in C([0, \infty), [0, \infty))$ with

$$f^{[n]}(s) > 0, \quad \forall s \in (0, \infty), \qquad \left(f^{[n]}\right)_0 = nf\left(\frac{1}{n}\right) > 0. \tag{41}$$

By (C3), it follows that

$$\lim_{n \to \infty} \left(f^{[n]}\right)_0 = \infty. \tag{42}$$

Now let us consider the auxiliary family of the equations

$$\begin{gathered} u'(t) = a(t) g(u(t)) u(t) - \lambda h(t) f^{[n]}(u(t - \tau(t))), \\ mt \in \mathbb{R}, \ u \text{ is } \omega\text{-periodic}. \end{gathered} \tag{43}$$

Let $\xi, \chi \in C[0, \infty)$ be such that

$$\begin{aligned} f^{[n]}(s) &= \left(f^{[n]}\right)_0 s + \xi(s) = nf\left(\frac{1}{n}\right) s + \xi(s), \\ g(s) &= g(0) + \chi(s). \end{aligned} \tag{44}$$

Note that

$$\lim_{s \to 0^+} \frac{\xi(s)}{s} = 0, \qquad \lim_{s \to 0^+} \chi(s) = 0. \tag{45}$$

Define a linear operator $A : D(A) \subset X \to X$

$$(Au)(t) = -u'(t) + a(t) g(0) u(t), \quad u \in D(A), \tag{46}$$

with

$$D(A) = \left\{u \in C^1[0, \omega] \mid u \text{ is } \omega\text{-periodic.}\right\}. \tag{47}$$

From [18], it follows that $A^{-1} : X \to X$ is compact and continuous.

Now (43) can be rewritten to the form

$$\begin{aligned} u(t) = {} & \lambda A^{-1}\left[h(\cdot)\left(f^{[n]}\right)_0 u(\cdot - \tau(\cdot))\right](t) \\ & + A^{-1} N(\lambda, u(\cdot))(t), \end{aligned} \tag{48}$$

where

$$N(\lambda, u(\cdot))(t) := [a(\cdot)\chi(u(\cdot))u(\cdot) - \lambda h(\cdot)\xi(u(\cdot - \tau(\cdot)))](t). \quad (49)$$

It is easy to check that

$$\lim_{\|u\| \to 0} \frac{N(\lambda, u)}{\|u\|} = 0, \quad \text{uniformly on bounded } \lambda \text{ intervals.} \quad (50)$$

It is easy to check that (48) is equivalent to

$$Au = \lambda h(\cdot)\left(f^{[n]}\right)_0 u(\cdot - \tau(\cdot))(t) + N(\lambda, u(\cdot))(t). \quad (51)$$

Let us consider (48) as a bifurcation problem from the trivial solution $u \equiv 0$.

Since $(f^{[n]})_0 > 0$, the results of Nonlinear Krein-Rutman Theorem (see Dancer [22] and Zeidler [23, Corollary 15.12]) for (48) can be stated as follows: there exists a continuum $\mathscr{C}_+^{[n]}$ of positive solutions of (48) joining $(\lambda^\diamond(g(0))/(f^{[n]})_0, 0)$ to infinity in K. Moreover, $\mathscr{C}_+^{[n]} \setminus \{(\lambda^\diamond(g(0))/(f^{[n]})_0, 0)\} \subset \operatorname{int} K$ and $(\lambda^\diamond(g(0))/(f^{[n]})_0, 0)$ is the only bifurcation point of (48) lying on trivial solutions line $u \equiv 0$.

Proof of Theorem 6. Let us verify that $\{\mathscr{C}_+^{[n]}\}$ satisfies all of the conditions of Theorem 2.

It follows from (42) that $\lambda^\diamond(g(0))/(f^{[n]})_0 =: \eta_n \to 0$ as $n \to \infty$. Therefore, (A2) holds.

Let $r > 0$ be fixed. Then there exists $n_0 \in \mathbb{N}$, such that

$$\frac{1}{n} \le \delta r. \quad (52)$$

Thus

$$M_r(f) = M_r\left(f^{[n]}\right), \quad n \ge n_0. \quad (53)$$

From this and Lemma 9, it follows that there exists $\lambda_*(r)$ with $0 < \lambda_*(r) < r(1-\sigma^l)/M_r \int_0^\omega h(s)ds$, such that (48) has no solution (λ, u) with

$$\|u\| = r, \quad 0 < \lambda < \lambda_*(r). \quad (54)$$

Since r is arbitrary, we see that (A1) is satisfied.

(A3) can be deduced directly from the Arzela-Ascoli Theorem and the definition of $f^{[n]}$.

Therefore, the superior limit of $\{\mathscr{C}_+^{[n]}\}$ contains an unbounded component $\mathscr{C}$ with $(0,0) \in \mathscr{C}$.

Moreover, $\mathscr{C} \cap \{(\mu, y) \mid 0 < \mu \le \lambda_*(r), \|y\| = r\} = \emptyset$.

Finally, we show that $\mathscr{C}$ meets $(0, \mathbf{0})$ and $(0, \infty)$.

Let $\{\mu_n, y_n\} \subset \mathscr{C}$ with

$$|\mu_n| + \|y_n\| \longrightarrow \infty. \quad (55)$$

We claim that $\mu_n \to 0^+$.

Assume on the contrary that $\mu_n \to +\infty$. Let

$$v_n(t) = \frac{y_n(t)}{\|y_n\|}. \quad (56)$$

Then

$$v_n'(t) = a(t)g(y_n(t))v_n(t) - \mu_n h(t) \times \frac{f(y_n(t-\tau(t)))}{y_n(t-\tau(t))} v_n(t-\tau(t)), \quad t \in \mathbb{R}, \quad (57)$$

$$v_n \in X. \quad (58)$$

So $v_n'(t) < 0$ on $[0,\omega]$ as $n \to \infty$. This contradicts (58). Thus (55) implies that

$$\|y_n\| \longrightarrow \infty. \quad (59)$$

Assume on the contrary that $\mu_n \to \alpha > 0$ (after taking a subsequence and relabeling if necessary).

Since $y_n(t) \ge \delta\|y_n\|$ on $[0,\omega]$, it follows from (59) that $y_n(t) \to \infty$ uniformly on $[0,\omega]$. This together with $f_\infty = \infty$ implies $v_n'(t) < 0$ on $[0,\omega]$ as $n \to \infty$. This contradicts (58) again. □

Theorem 10. *Let (C1)-(C2) hold. Assume that*

(C4) $f_0 = 0$; $f_\infty = 0$.

Then Σ contains a component $\mathscr{C}$ satisfying the following.

(1) *For given $\hat{b} > 0$,*

$$\sup\left\{\lambda \mid (\lambda, u) \in \left(\mathscr{C} \cap \Omega_{\hat{b},1}\right)\right\} = +\infty. \quad (60)$$

(2) *For every $r > 0$, there exists $\eta(r) > 0$, such that*

$$\mathscr{C} \cap \{(\mu, u) \in [\mu(r), \infty) \times X \mid \|u\| = r\} = \emptyset. \quad (61)$$

Corollary 11. *Let (C1)-(C2) and (C4) hold. Then there is a constant $\eta^* > 0$ such that* (24) *has at least two positive solutions as $\lambda > \eta^*$.*

To prove Theorem 10, we define $f^{[n]}$ as in (40). Notice that (C4) implies that

$$\lim_{n\to\infty}\left(f^{[n]}\right)_0 = 0. \quad (62)$$

Let ξ, χ be the function satisfying (44)-(45).

Now (43) can be rewritten to the form

$$u(t) = \lambda A^{-1}\left[h(\cdot)\left(f^{[n]}\right)_0 u(\cdot - \tau(\cdot))\right](t) + A^{-1}N(\lambda, u(\cdot))(t), \quad (63)$$

where

$$N(\lambda, u(\cdot))(t) := [a(\cdot)\chi(u(\cdot))u(\cdot) - \lambda h(\cdot)\xi(u(\cdot - \tau(\cdot)))](t). \quad (64)$$

It is easy to check that

$$\lim_{\|u\| \to 0} \frac{N(\lambda, u)}{\|u\|} = 0, \quad \text{uniformly on bounded } \lambda \text{ intervals.} \quad (65)$$

Let us consider (63) as a bifurcation problem from the trivial solution $u \equiv 0$.

By similar method to deal with (48) in which $f_0 = \infty$, we have from $(f^{[n]})_0 > 0$ and [23, Corollary 15.12]) that there exists a continuum $\mathscr{C}_+^{[n]}$ of positive solutions of (63) joining $(\lambda^\diamond(g(0))/(f^{[n]})_0, 0)$ to infinity in K. Moreover, $\mathscr{C}_+^{[n]} \setminus \{(\lambda^\diamond(g(0))/(f^{[n]})_0, 0)\} \subset \operatorname{int} K$ and $(\lambda^\diamond(g(0))/(f^{[n]})_0, 0)$ is the only bifurcation point of (48) lying on trivial solutions line $u \equiv 0$.

Lemma 12. *Assume that (C1)-(C2) hold. If $u \in \partial K_r$, $r > 0$, then*

$$\|T_\lambda u\| \geq \lambda \frac{\sigma^L \widehat{m}_r}{1 - \sigma^L} \int_0^\omega h(s)\, ds, \tag{66}$$

where

$$\widehat{m}_r = \min_{\delta r \leq x \leq r} \{f(x)\}. \tag{67}$$

Proof. Since $\sigma^L/(1-\sigma^L) \leq G_u(t,s)$ and $f(u(t)) \geq \widehat{m}_r$ for $t \in [0,\omega]$, it follows that

$$\begin{aligned}\|T_\lambda u\| &\geq \lambda \int_t^{t+\omega} G_u(t,s) h(s) f(u(s))\, ds \\ &\geq \lambda \frac{\sigma^L \widehat{m}_r}{1-\sigma^L} \int_t^{t+\omega} h(s)\, ds \geq \lambda \frac{\sigma^L \widehat{m}_r}{1-\sigma^L} \int_0^\omega h(s)\, ds.\end{aligned} \tag{68}$$

□

Lemma 13. *Assume that (C1)-(C2) and (C4) hold, and let I be a compact subinterval of $(0,\infty)$. Then*

$$\|u\| \leq M_I, (\lambda, u) \in \left\{(\lambda, u) \in C_+^{[n]} \mid \lambda \in I\right\} \tag{69}$$

for some positive constant M_I, independent of n, λ, and u.

Proof. Assume on the contrary that there exists a sequence $\{(\mu_k, y_k)\} \subset \mathscr{C}_+^{[n]} \cap (I \times K)$ such that

$$\|y_k\| \longrightarrow \infty. \tag{70}$$

Set $v_k(t) = y_k(t)/\|y_k\|$. Then

$$\|v_k\| = 1, \tag{71}$$

$$\begin{aligned}v_n'(t) &= a(t) g(y_n(t)) v_n(t) - \mu_n h(t) \\ &\quad \times \frac{f(y_n(t-\tau(t)))}{y_n(t-\tau(t))} v_n(t-\tau(t)), \quad t \in \mathbb{R},\end{aligned} \tag{72}$$

$$v_n \in X. \tag{73}$$

By (72), $\{v_n'\}$ is bounded in X. This together with the fact that $\{\mu_n\} \subset I$ implies that there exists $(\mu_*, v_*) \in I \times X$ with

$$\|v_*\| = 1, \tag{74}$$

such that

$$\lim_{k\to\infty} (\mu_k, v_k) = (\mu_*, v_*), \quad \text{in } \mathbb{R} \times X, \tag{75}$$

(after choosing a subsequence and relabeling if necessary). Since $\{g(u_n)\}$ is bounded in X, $\{g(u_n)\}$ is bounded in $L^2(0,\omega)$, and subsequently, $g(u_n) \rightharpoonup \widehat{g}$ for some $\widehat{g} \in L^2(0,\omega)$. By the standard method, we can prove that

$$l \leq \widehat{g}(t) \leq L, \quad \text{a.e. } t \in [0,\omega]. \tag{76}$$

Moreover, combining (75) and (76) with the assumption $f_\infty = 0$ and the corresponding integral equations of (72) and (73) and using Lebesgue dominated convergence theorem, we conclude that

$$v_*'(t) - a(t) \widehat{g}(t) v^* = 0, \quad \text{a.e. } t \in \mathbb{R}, \tag{77}$$

$$v_* \in X. \tag{78}$$

Note that $a(t)\widehat{g}(t)v^* \geq 0$ and $a(t)\widehat{g}(t)v^* \not\equiv 0$ on $[0,\omega]$. This means that $v_*(0) < v_*(\omega)$. However, this contradicts (78). □

Now, we are in the position to prove Theorem 10.

Proof of Theorem 10 (sketched). Fixed $r > 0$. From Lemma 12, it follows that (63) has no solution if

$$\lambda > \lambda^*(r) =: 1 + \frac{r}{(\sigma^L \widehat{m}_r/(1-\sigma^L)) \int_0^\omega h(s)\, ds}. \tag{79}$$

Applying Lemma 13, it is easy to verify that $\{\mathscr{C}_+^{[n]}\}$ satisfies all of the conditions of Theorem 3. So, there exists a component $\mathscr{C}$ in $\overline{\lim}\, \mathscr{C}_+^{[n]}$ such that

(a) both $\mathscr{C} \cap \Omega_{\lambda^*(1),1}$ and $(\mathscr{C} \cap (([a,\infty) \times X) \setminus \Omega_{\lambda^*(1),1}))$ are unbounded;

(b) $\mathscr{C} \cap \{(\mu, y) \mid \mu \geq \lambda^*(1), \|y\| = 1\} = \emptyset$.

Lemma 13 guarantees that $\mathscr{C}$ satisfies

$$\sup\{\lambda \mid (\lambda, u) \in (\mathscr{C} \cap \Omega_{\lambda^*(1),1})\} = +\infty. \tag{80}$$

□

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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